

Very Simple Tight Bounds on the Q-Function

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Abstract – This paper presents realization of the paper [1] contains calculation of new lower and upper bounds on the Gaussian Q-function, which are in simple algebraic form that contain only two exponential terms. Lower and upper bounds are obtained from selected coefficients which are stated in the paper [1]. Moreover, this new lower and upper bounds allow Q-function to calculate simpler than numeric integration. In addition to realization of paper [1], combination of new Supertight bounds and its performance and accuracy are represent in this paper with comparing other methods such as numeric integration, Jensen-Cotes[2] methods and exponential representation [3].

Index Terms-Gaussian Q-function, Super-Tight Bounds, BER, BPSK, Communication

I. INTRODUCTION

Q-function which is used for error analysis in communication systems over fading channels has an integral form. Q-function is very important for communication systems because of that error probability of communication channels is calculated by the help of this function. It is needed to determine error behavior of channel to achieve better quality in communication. However, it takes a long time to calculate Q-function because of integral form of the function. So that, we focused on how to calculate Q-function having better performance and more speedy.

In this study, new lower and upper bounds on the Gaussian Q-function are obtained on Matlab. As part of the project, new bounds of Q function is calculated with new method which contains only two exponential terms with a constant and a rational coefficient. The curve of obtained Q-function and the other methods such as Jensen Cotes, exponential, numeric integration are plotted to present correctness of new method. At the second step of the study, bit error rate (BER) of QPSK modulation on AWGN channel is calculated by the help of new Gaussian Q-function. Results are compared with the theoretical bit error rate curve and Monte Carlo simulation curve of BPSK modulation.

II. THE NEW BOUNDS ON $Q(x)$

The Q-function is a convenient way to express probabilities for Gaussian random variables.

A. The formal expression of Q-function

$$Q(x) = \frac{1}{\sqrt{2\pi}} \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{t^2}{2}\right) dt, \quad (1)$$

B. Under condition of $x \geq 0$

In the reference paper [1], Q-function and parameters for bounds is described in below.

$$Q_B(x; a, b, n) = Q_B^n(x; a, b), \quad (2a)$$

$$= \sum_{k=0}^n \binom{n}{k} \frac{\exp\left(-\left(n-\frac{k}{2}\right)x^2\right)}{a^{(n-k)} b^k (x+1)^k}, \quad (2b)$$

$$Q_B(x; a, b) = \frac{\exp(-x^2)}{a} + \frac{\exp(-x^2/2)}{b(x+1)}, \quad (3)$$

For all $x \geq 0$ and $n \in \mathbb{N}$

$$Q_B(x; a_L, b_L, n) \leq Q^n(x) \leq Q_B(x; a_U, b_U, n), \quad (4)$$

Where a_U and b_U satisfy

$$a_U \geq (98 + 18\sqrt{17}) \exp\left(\frac{\sqrt{17}-9}{4}\right) - 2 \approx 48.8828, \quad (5)$$

$$0 < b_U \leq \frac{3a_U\sqrt{2\pi}}{4a_U - 8\sqrt{2\pi}\exp(-0.5)} \approx 2.0047, \quad (6)$$

$$b_L \geq \sqrt{2\pi} \approx 2.5066, \quad (7)$$

$$0 < a_L \leq \frac{8b_L\sqrt{2\pi}\exp(-0.5)}{4b_L - 3\sqrt{2\pi}} \approx 12.1628, \quad (8)$$

Considering $Q(-x) = 1 - Q(|x|)$, expression of Q-function can be written as below;

$$Q(x; a_L, b_L) \leq Q(x) \leq Q(x; a_U, b_U), \quad (9)$$

$$\frac{\exp(-x^2)}{a_L} + \frac{\exp(-x^2/2)}{b_L(x+1)} \leq Q(x) \leq \frac{\exp(-x^2)}{a_U} + \frac{\exp(-x^2/2)}{b_U(x+1)}, \quad (10)$$

It is clear that Q-function always gets value between upper and lower bounds. Upper and lower bound expression is given by equation (11a) and (11b).

$$Q_L(x; a_L; b_L) = \frac{\exp(-x^2)}{a_L} + \frac{\exp\left(-\frac{x^2}{2}\right)}{b_L(x+1)}, \quad (11a)$$

$$Q_U(x; a_U; b_U) = \frac{\exp(-x^2)}{a_U} + \frac{\exp\left(-\frac{x^2}{2}\right)}{b_U(x+1)}, \quad (11b)$$

In addition of new upper and lower bound equations of Q-function, bounds are combines with two exponential. The new formula is given by equation (12) with the parameters of $d = 11$ on equation (13).

$$Q_c(x; d, a_L, b_L, a_U, b_U) = \left[\left(\frac{e^{-dx}}{2} \right) Q_L(x; a_L, b_L) \right] + \left[\left(1 - \frac{e^{-dx}}{2} \right) Q_U(x; a_U, b_U) \right] \quad (12)$$

$$d = 11, \quad (13)$$

The new Q-function bounds which can be seen on equation (10) is represented by only two exponential components and its combined version has extra two exponential component. It is easily said that the new simple form of Q-function is calculated faster than the original form.

III. ACCURACY ANALYSIS AND COMPARISONS

In this part of paper, new method in equation (12) and other calculation methods is compared in two topic which are accuracy and calculation performance.

A. Accuracy Analysis

In the Figure 1, it can be observed comparison of exponential expression for $N=2$ [3], numerical integration, new Supertight bounds [1] and new combined form on (12).

Numerical integration is the most accurate values because of calculated using formal formula of Q-function.

In the reference paper [3], the exponential form of Q-function as represented in (14).

$$Q_E(x; N) = \sum_{k=1}^N \frac{\exp\left(-\frac{\left(\frac{x}{\sqrt{2}}\right)^2}{\sin\left(\frac{k\pi}{(2N)^2}\right)}\right)}{2N}, \quad (14)$$

$N=2$ is chosen for comparison. By rising value of N , Q_E becomes more close to true values but its calculation times rises.

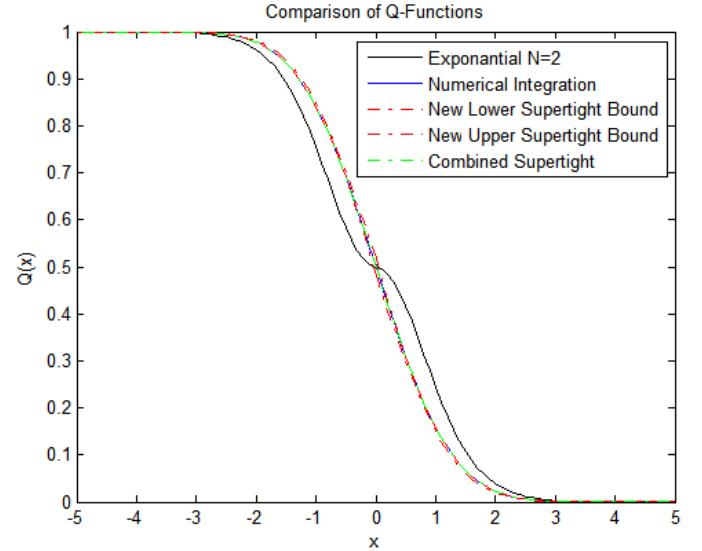


Figure 1: Comparison of Q-functions

It can be seen that in the figure 1, New Supertight bounds and its combined form is so close to numerical integration result but exponential expression form is not close enough. It is possible to make closer by increasing N .

Another popular approaching method to Q-function is Jensen-Cotes method. In the paper [2], upper and lower bounds are represented in equation (16a) and (16b)

$$f_1 = \sqrt{x^2 + 3 - \sqrt{(x^2 - 1)^2 + 8}}, \quad (15a)$$

$$f_2 = \sqrt{4 - f_1^2}, \quad (15b)$$

$$Q_U^J(x) = 3f_1 \frac{\exp\left(-\frac{2x^2}{2-f_2}\right)}{4\pi+f_2\pi} + (4 + (\pi - 2)f_2 - 2f_1) \frac{\left(\exp\left(-\frac{x^2}{2}\right) + \exp\left(-\frac{2x^2}{f_1^2}\right)\right)}{16+4f_2(\pi-2)} \quad (16a)$$

$$Q_U^J = 0.25 \exp\left(-\frac{x^2}{2}\right) + \frac{\exp\left(-\frac{x^4}{2x^2+1}\right)(1-x+\sqrt{x^2+4})}{2(1+\sqrt{x^2+4})}, \quad (16b)$$

Numerical integration, exponential expression, Jensen-Cotes bounds and New Supertight bounds are compared in figure 2. In the graph, It is seen that The New Upper Supertight bound has exactly same value with numerical integration in interval between $x=2$ and $x=4$ and the New Lower Supertight bound is very close to them.

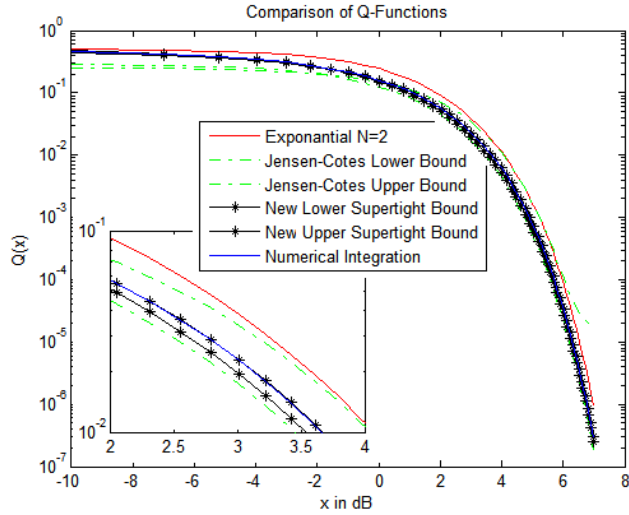


Figure 2: Comparison of Q-functions(x in dB)

Other bounds in figure 2 have worse approximation to numerical integration results. It is very clear that the new methods is given by formula (11a) and (11b) in the reference paper [1] is supertight bounds on Q-function. Using combination of these supertight bounds within a formula (12), the new simple Q-function is formed.

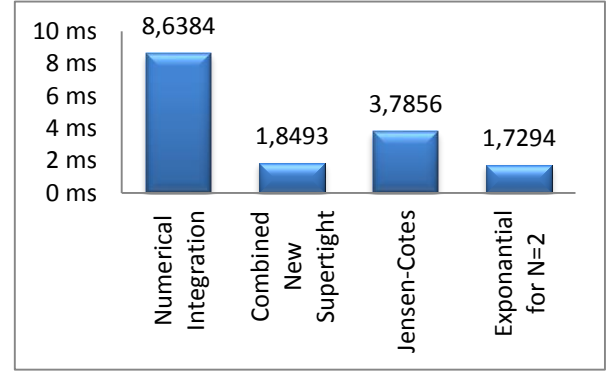
B. Performance analysis

In the previous section accuracies of methods are compared and in this section performances analysis of methods are represented.

Calculation time performance analysis of four methods is represented in table 1. Results have been brought out with running MATLAB codes of methods 1000 times. Every result block indicates average calculation time of each methods in milliseconds.

According the performance analysis, average calculation time of New Combined methods is nearly 4.7 times faster than numerical integration. Although Exponential for $N=2$ performance is nearly 5 times faster than numerical integration, its accuracy performance is very bad. It is trade of between performance and accuracy for exponential method.

TABLE I. PERFORMANCE ANALYSIS



As a result of accuracy and performance analysis, the New Combined Supertight method is the most effective solution to calculate Q-function with high performance and accuracy.

IV. IMPLEMENTATION

On the implementation part of the project, Bit Error Rate (BER) which is the of Binary Phase Shifting Keying (BPSK) Modulation System with AWGN channel is calculated by the help of combined New Supertight Bounds Q-function to present reliability of it.

Bit error rate is the percentage of bits that have errors relative to the total number of bits received in a communication system. In the communication systems with noisy channel, Bit Error Rate (BER) may be expressed as a form of Q-function.

In the communication system with noisy channel, for instance BPSK modulation system with AWGN channel, received signal can be written as:

$$y = x + n \quad (17)$$

For BPSK modulation, for $x \in \{-A, A\}$, $n \sim N(0, \sigma^2)$ and $\sigma^2 = N_0$, $d_{min} = 2A$ is defined as;

$$\frac{E_b}{N_0} = \frac{d_{min}^2}{4N_0}, \quad (18)$$

So bit error probability is calculated that:

$$P_b = P\{n > A\} = \int_A^\infty \frac{\exp\left(\frac{-x^2}{2\sigma^2/2}\right)}{\sqrt{2\pi\sigma^2/2}}, \quad (19)$$

$$P_b = Q\left(\sqrt{\frac{d_{min}^2}{2N_0}}\right) = Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) = Q(\sqrt{2\gamma}), \quad (20)$$

BER may be estimated by Monte Carlo Simulation which is a form of computational algorithms that rely on repeated

random sampling to compute results. BER of the BPSK Modulation System on MATLAB Simulink is calculated Monte Carlo Simulation theoretically

In this project, to prove correctness of new supertight bounds, BER curve of BPSK Modulation is calculated by the help of the new method. At the same time, theoretical BER and BER of Monte Carlo Simulation are calculated by using Bertool in MATLAB. All methods are plotted at the same graph.

The curve of new supertight bound is checked against other methods. So it has seen that, the curves are at the same line.

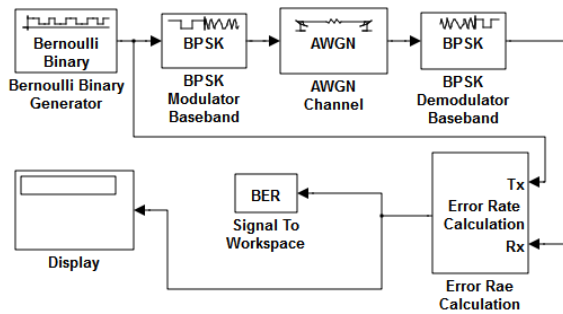


Figure 3: Block Diagram of BPSK modulation on Matlab-Simulink

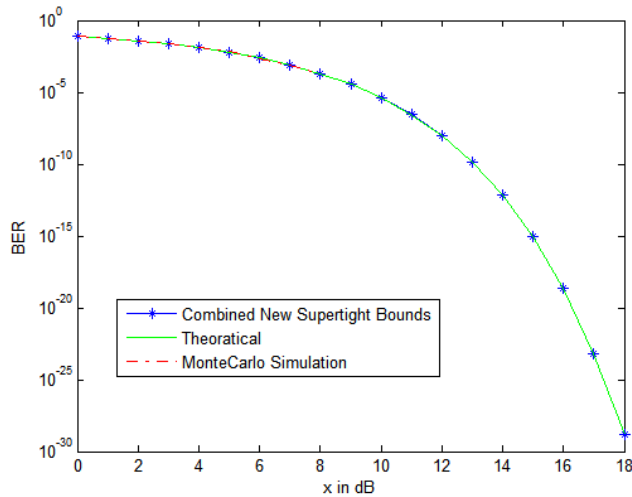


Figure 4: Comparison of BER curves are obtained with using different methods

V. CONCLUSIONS

In this study, new and simple bounds of Q-function are calculated and compared with other methods such as Jensen-Cotes, exponential form, numerical integration.

Because of the fact that Q-function has an integral form, calculating Q-function takes a long time and shows worse performance. The New Supertight bounded Q-function has an exponential form so it represents better performance. Mathematical operations take shorter time than other methods.

The curves of New Supertight Lower Bound and New Supertight Upper Bound are plotted and checked against other methods. The curve of new Supertight Bounds Q-function is similar to others in addition to this, speed performance is much better.

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